Supersonic Wave Drag of Planar Singularity Distributions

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A closed-form expression for the supersonic wave drag of a general planar source-doublet distribution is obtained using the linearized differential equation and exact nonlinear boundary conditions. The result is useful for drag computations and analytic drag optimization studies.

Introduction

THIS paper derives a planar counterpart to von Kármán's[1] well-known drag formula for axisymmetric lineal source distributions but which includes the effect of doublets. It provides an explicit evaluation of the author's[2] formal results (for general nonplanar singularity distributions) in the planar case. Current wave drag prediction methods in supersonic aerodynamics are based on the classical analyses of von Kármán and Hayes.[3] An exact result due to Hayes applies only to the source problem and is summarized easily. For sources of density $\tilde{f}(Q)$, where Q is the source coordinate, define an equivalent density f such that

$$f(x_i;\theta)\mathrm{d}x_i = \iiint_{V(x_i;\theta)} \tilde{f}(Q)\,\mathrm{d}V \tag{1}$$

Here θ is an angle measured in a plane normal to the freestream, and $V(x_i;\theta)$ is the region contained between two Mach planes $x_i = x - \beta y\cos\theta_0 - \beta z\sin\theta_0$ perpendicular to a given meridian plane $\theta = \theta_0$ and intersecting the x axis at $x = x_i$ and $x = x_i + \mathrm{d}x_i$ (x is aligned with the undisturbed flow at infinity and $\beta = \sqrt{M_\infty^2 - 1}$, where M_∞ is the freestream Mach number). Invariance arguments suggest local application of von Kármán's drag formula, that is,

$$\frac{\mathrm{d}D_w}{\mathrm{d}\theta} = -\frac{\rho_\infty U_\infty^2}{8\pi^2}\int_0^\ell \int_0^\ell f'(x_1;\theta)f'(x_2;\theta)\log_e|x_1 - x_2|\,\mathrm{d}x_1\mathrm{d}x_2 \tag{2}$$

where ρ_∞ is the undisturbed density, U_∞ is the speed at infinity, and ℓ is the body length. The net wave drag D_w then is obtained by integration, giving

$$D_w = \int_0^{2\pi}\left(\frac{\mathrm{d}D_w}{\mathrm{d}\theta}\right)\mathrm{d}\theta \tag{3}$$

To amend these results for nonthickness effects, Hayes included the influence of elementary horseshoe vortices such as were used in the classical theory to represent lateral force components such as lift and sideforce. A function $\tilde{h}$ was defined such that

$$\tilde{h} = \tilde{f} - \beta(\tilde{\ell}\sin\theta + \tilde{S}\cos\theta) \tag{4}$$

where $\rho_\infty U_\infty^2 \tilde{\ell}$ and $\rho_\infty U_\infty^2 \tilde{S}$ are lift and side forces per unit volume. As before, introduce

$$h(x_i;\theta)\mathrm{d}x_i = \iiint_{V(x_i;\theta)} \tilde{h}(Q)\,\mathrm{d}V \tag{5}$$

Then, Eqs. (2) and (3) hold with f replaced by h. However, the foregoing results contain certain implied near-planar body assumptions and are somewhat restrictive. They arise in considering the far-field momentum flux due to distributions of horseshoe vortices, which are restricted to lie along surfaces whose generators are aligned with the streamwise axis of the governing differential equation. Thus, the preceding equations bear the limiting relationships connecting local source strength to local frontal area change and local vorticity strength to local lift. Because recent advances in computational panel methodology[4] now enable the modeling of completely general nonplanar aerodynamic configurations with surface singularity distributions, the need for improved wave drag prediction methods was stimulated. This led to a recent extension of Hayes' formal algorithm to cover arbitrary source/doublet distributions on general nonplanar surfaces by the present author.[2] In that analysis, the linearized differential equation was used, along with exact nonlinear surface boundary conditions. (Solutions to such problems are "second-order solutions."[5]) The exact wave drag for this problem, it turned out, required only simple modifications to Hayes' method. We assume known R, P, and Q, the strengths of doublets whose axes are aligned in the directions of the x, z, and y axes, respectively. (An arbitrary doublet axis orientation is simply a linear combination of these three components.) The perturbation velocity potential $\varphi = \varphi_S + \varphi_D$ consists of two parts, φ_S due to the source distribution and φ_D due to the doublet distribution, given by

$$\varphi_S(x,y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{C(y_1)} \frac{\tilde{f}(x_1,y_1)}{[(x-x_1)^2 - \beta^2(y-y_1)^2 - \beta^2(z-z_1)^2]^{1/2}} \times G_1(x_1,y_1)\,\mathrm{d}x_1\mathrm{d}y_1 \tag{6}$$

and

$$\varphi_D(x,y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{C} \frac{\mu(x_1,y_1;x,y,z)\,G_1\mathrm{d}x_1\mathrm{d}y_1}{[(x-x_1)^2 - \beta^2(y-y_1)^2 - \beta^2(z-z_1)^2]^{3/2}} \tag{7}$$

In Eqs. (6) and (7), note that

$$\mu(x_1,y_1;x,y,z) = R(x_1,y_1)(x-x_1) + \beta^2 P(x_1,y_1)[z - z_1(x_1,y_1)] + \beta^2 Q(x_1,y_1)(y-y_1)$$

C is defined by the solution $x_1 = C(y_1)$ to

$$(x-x_1)^2 - \beta^2(y-y_1)^2 - \beta^2[z - z_1(x_1,y_1)]^2 = 0$$

Received June 17, 1977; revision received Jan. 11, 1978.
Index category: Supersonic and Hypersonic Flow.

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The quantity $\tilde{f}(x_1,y_1)$ is the areal source strength, and

$$G_1(x_1,y_1)\,dx_1dy_1 = (1+Z_{1_{x_1}}^2+Z_{1_{y_1}}^2)^{1/2}\,dx_1dy_1 = dS$$

is an incremental surface area for $Z_1=Z_1(x_1,y_1)$. The drag due to sources alone is given exactly by Eqs. (1-3). But the total wave drag is not given by Eq. (4), since it implicitly assumes a near-planar body. It is given exactly by Eq. (5), though, provided that we replace Eq. (4) by the more complete expression given in Eq. (8). For thin, near-planar bodies, Eq. (8) reduces to Eq. (4). This limit refers to $Z_1=0$, $G_1=1$, and $R=0$, so that Eq. (4) is recovered; the term P_{x_1} is proportional to the lift density $\tilde{\ell}$, whereas Q_{x_1} is proportional to the side-force density $\tilde{S}$:

$$\tilde{h}=\tilde{f}+\frac{\begin{bmatrix}\beta^2\sin\theta\cos\theta(Q_{x_1}G_1z_{1_{x_1}}+QG_{1_{x_1}}z_{1_{x_1}}-QG_1z_{1_{x_1x_1}})-(R_{x_1}G_1+RG_{1_{x_1}})-\beta\sin\theta(P_{x_1}G_1+PG_{1_{x_1}})-\beta\cos\theta(Q_{x_1}G_1+QG_{1_{x_1}})\\ +\beta\sin\theta(R_{x_1}G_1z_{1_{x_1}}+RG_{1_{x_1}}z_{1_{x_1}}-RG_1z_{1_{x_1x_1}})+\beta^2\sin^2\theta(P_{x_1}G_1z_{1_{x_1}}+PG_{1_{x_1}}z_{1_{x_1}}-PG_1z_{1_{x_1x_1}})\end{bmatrix}}{(1-\beta\sin\theta z_{1_{x_1}})^2G_1(x_1,y_1)} \tag{8}$$

Analysis

The foregoing generalization to Hayes' formal numerical procedure requires only simple modifications to existing computer codes widely in use. However, in many practical applications, especially analytical ones, it is desirable to proceed directly from a closed-form integral expression. For example, we can point to the well-known use of von Kármán's drag formula (for lineal source distributions) in parametric optimization studies. For areal distributions, progress in terms of general results has been impeded mainly because of geometric difficulties in describing the far-field momentum flux vector, even for simple shapes. However, for planar source-doublet distributions, it turns out that the wave drag integrand can be given in closed form, and it is this evaluation that forms the subject of the present paper. The results are based on the linearized differential equation but without the classical application of linearized boundary conditions on mean surfaces. The boundary-value problem considered therefore corresponds to an exact solution of the zeroth-order Rayleigh-Janzen expansion. The results, again, apply to general nonplanar configurations representable by (given) planar singularity distributions; an example might be a Rankine body with an intersecting wing.

The wave drag of a planar ($Z_1=0$) surface distribution of sources of areal strength $f(x_1,y_1)$ and doublets of strength $P(x_1,y_1)$ oriented in the z direction can be determined by enclosing the finite body by a surface S. The net force acting on it then can be obtained by examining the momentum flux across S. For linearized supersonic flow, one can write

$$D_w=-\rho_\infty U_\infty^2\int\int_{S_2}\phi_x\phi_r\,dS_2 \tag{9}$$

where S_2 is a circular control volume that encloses the singularity distribution and is aligned with the flow direction. Now enclose the body with two Mach cones (as shown in Fig. 1), and define the origin (0,0,0) by the left vertex, and the right vertex by the point (ℓ,0,0). (Note that the wave drag contributions from S_1 and S_3 are zero; also, one must add to D_w the vortex drag of the infinite wake in assessing the total inviscid drag.) Now if $\varphi=\varphi_S+\varphi_{D(Z)}$, and R is the radius of S_2, we have

$$\mathfrak{D}\equiv-\frac{D_w}{\rho_\infty U_\infty^2}=\int_{\theta=0}^{\theta=2\pi}\int_{x=\beta R}^{x=\ell_3}(\phi_x\phi_r)_{r=R}R\,dx\,d\theta=\mathfrak{D}_1+\mathfrak{D}_2+\mathfrak{D}_3+\mathfrak{D}_4 \tag{10}$$

where ℓ_3 is, as yet, an unspecified distance far downstream. The $\mathfrak{D}$'s in Eq. (10) refer to, respectively, the products $(\partial\varphi_s/\partial x)$ $(\partial\varphi_s/\partial r)$, $(\partial\varphi_D/\partial x)$ $(\partial\varphi_D/\partial r)$, $(\partial\varphi_s/\partial x)$ $(\partial\varphi_D/\partial r)$, and $(\partial\varphi_D/\partial x)$ $(\partial\varphi_s/\partial r)$. Here $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are drags due to sources and doublets, respectively, and $\mathfrak{D}_3$ and $\mathfrak{D}_4$ are interference drags. Although Eq. (10) is valid with any value of R, geometric simplifications are possible by letting R tend to infinity. We shall carry through in detail the evaluation of the drag due to sources. We first write φ_S in the form

$$\phi_S=-\frac{1}{2\pi}\int_{y_1=-\infty}^{y_1=\infty}\int_{x_1=-\infty}^{x_1=x-\beta\sqrt{r^2+y_1^2-2ry_1\cos\theta}}\frac{f(x_1,y_1)}{\sqrt{(x-x_1)^2-\beta^2(r^2+y_1^2-2ry_1\cos\theta)}}\,dx_1dy_1 \tag{11}$$

If $f(-\infty,y_1)=0$, integration by parts and differentiation lead to

$$\frac{\partial\phi_s}{\partial x}=-\frac{1}{2\pi}\int_{y_1=-\infty}^{y_1=\infty}\int_{x_1=-\infty}^{x_1=x-\beta\sqrt{r^2+y_1^2-2ry_1\cos\theta}}\frac{f_{x_1}(x_1,y_1)}{\sqrt{(x-x_1)^2-\beta^2(r^2+y_1^2-2ry_1\cos\theta)}}\,dx_1dy_1 \tag{12}$$

$$\frac{\partial\phi_s}{\partial r}=\frac{1}{2\pi}\int_{y_1=-\infty}^{y_1=\infty}\int_{x_1=-\infty}^{x_1=x-\beta\sqrt{r^2+y_1^2-2ry_1\cos\theta}}\frac{(r-y_1\cos\theta)(x-x_1)f_{x_1}(x_1,y_1)\,dx_1dy_1}{(r^2+y_1^2-2ry_1\cos\theta)\sqrt{(x-x_1)^2-\beta^2(r^2+y_1^2-2ry_1\cos\theta)}} \tag{13}$$

Now in calculating $\mathfrak{D}_1$ we retain the subscripts 1 in Eq. (12) while using dummy subscripts 2 in Eq. (13). Then the integral over products, that is,

$$-4\pi^2\mathfrak{D}_1=\int_{\theta=0}^{\theta=2\pi}\int_{x=\beta R}^{x=\ell_3}\left(\int_{y_1=-\infty}^{y_1=\infty}\int_{x_1=-\infty}^{x_1=x-\beta\sqrt{R^2+y_1^2-2Ry_1\cos\theta}}\frac{f_x(x_1,y_1)\,dx_1dy_1}{\sqrt{(x-x_1)^2-\beta^2(R^2+y_1^2-2Ry_1\cos\theta)}}\right)$$
$$\left(\int_{y_2=-\infty}^{y_2=\infty}\int_{x=-\infty}^{x_2=x-\beta\sqrt{R^2+y_2^2-2Ry_2\cos\theta}}\frac{(R-y_2\cos\theta)(x-x_2)f_x(x_2,y_2)\,dx_2dy_2}{(R^2+y_2^2-2Ry_2\cos\theta)\sqrt{(x-x_2)^2-\beta^2(R^2+y_2^2-2Ry_2\cos\theta)}}\right)R\,dx\,d\theta \tag{14}$$

can be written as the multiple integral

$$-4\pi^2\mathfrak{D}_I = \int_{y_1=-\infty}^{y_1=\infty}\int_{y_2=-\infty}^{y_2=\infty}\int_{\theta=0}^{\theta=2\pi}\int_{x=\beta R}^{x=\ell_3}\int_{x_1=-\infty}^{x_1=x-\beta\sqrt{R^2+y_1^2-2Ry_1\cos\theta}}\int_{x_2=-\infty}^{x_2=x-\beta\sqrt{R^2+y_2^2-2Ry_2\cos\theta}}$$

$$\frac{(R-y_2\cos\theta)(x-x_2)Rf_x(x_1,y_1)f_x(x_2,y_2)\,dx_2dx_1dxd\theta dy_2dy_1}{(R^2+y_2^2-2Ry_2\cos\theta)\sqrt{(x-x_1)^2-\beta^2(R^2+y_1^2-2Ry_1\cos\theta)}\ \sqrt{(x-x_2)^2-\beta^2(R^2+y_2^2-2Ry_2\cos\theta)}} \tag{15}$$

since the outer four integrals contain constant limits. In the preceding expressions, the lower limits in the dx_1 and dx_2 integrations can be replaced by zero, by virtue of Fig. 1. For the upper limits, we recall that the singularity distribution is finite in extent. If at an observation point $P_0(x,y,z)$ we have $x-\beta R>\ell$, so that the singularity distribution lies wholly within the forecone emanating from P_0, then P_0 is influenced by all of the sources, and the upper limits can be taken over the entire body. For this, it is sufficient to choose $x_1=x_2=\ell$ whenever $x'=x-\beta R\geq\ell$. We therefore can write

$$-4\pi^2\mathfrak{D}_I = \int_{y_1=-\infty}^{y_1=\infty}\int_{y_2=-\infty}^{y_2=\infty}\int_{\theta=0}^{\theta=2\pi}\int_{x'=0}^{x'=\ell_3-\beta R}\int_{x_1=0}^{x_1=x'+\beta R-\beta\sqrt{R^2+y_1^2-2Ry_1\cos\theta};\ell}\int_{x_2=0}^{x_2=x'+\beta R-\beta\sqrt{R^2+y_2^2-2Ry_2\cos\theta};\ell}$$

$$\frac{(R-y_2\cos\theta)(\beta R+x'-x_2)Rf_x(x_1,y_1)f_x(x_2,y_2)\,dx_2dx_1dx'd\theta dy_2dy_1}{(R^2+y_2^2-2Ry_2\cos\theta)\sqrt{(\beta R+x'-x_1)^2-\beta^2(R^2+y_1^2-2Ry_1\cos\theta)}\ \sqrt{(\beta R+x'-x_2)^2-\beta^2(R^2+y_2^2-2Ry_2\cos\theta)}} \tag{16}$$

where, in the inner integrals, the upper limits are

$$x_{1,2}=x'+\beta R-\beta(R^2+y_{1,2}^2-2Ry_{1,2}\cos\theta)^{1/2} \quad\text{or}\quad x_{1,2}=\ell$$

depending on whether x' is less than or greater than ℓ. Now we evaluate the foregoing equation in the limit as R tends to infinity. According to the way the rear disk S_3 was chosen, ℓ_3 must be larger than R at all times. Hence, ℓ_3 approaches infinity as $R\to\infty$ such that $\beta R/\ell_3\ll 1$. As $\beta R\to\infty$, but $x'=x-\beta R$ remains finite, the integrand in Eq. (16) tends to

$$\frac{1}{2}\,\frac{f_x(x_1,y_1)f_x(x_2,y_2)}{\sqrt{x'-x_1+\beta y_1\cos\theta}\ \sqrt{x'-x_2+\beta y_2\cos\theta}} \tag{17}$$

whereas for $x'\gg\beta R$, which is the case for part of the integration region, a different limiting integrand is obtained. It thus is convenient to split the interval of integration over x' up into two parts, the first from 0 to a, and the second from a to $\ell_3-\beta R$. In the first integral, the expression given in Eq. (17) can be used, whereas in the second, since $a\gg\ell$, $x'\gg\beta R$, and $x_1,x_2\sim O(\ell)$, x_1 and x_2 can be neglected in comparison to x'. Thus in the limit of large βR, when the inner limits are also approximated to $O(1/\beta R)$, we have

$$-4\pi^2\mathfrak{D}_I = \int_{y_1=-\infty}^{y_1=\infty}\int_{y_2=-\infty}^{y_2=\infty}\int_{\theta=0}^{\theta=2\pi}\Bigg[\int_{x'=0}^{x'=a}\int_{x_1=0}^{x_1=x'+\beta y_1\cos\theta;\ell}\int_{x_2=0}^{x_2=x'+\beta y_2\cos\theta;\ell}\frac{f_x(x_1,y_1)f_x(x_2,y_2)\,dx_2dx_1dx'}{2\sqrt{x'-x_1+\beta y_1\cos\theta}\ \sqrt{x'-x_2+\beta y_2\cos\theta}}+\int_{x'=a\gg\ell}^{x'=\ell_3-\beta R}$$

$$\int_{x_1=0}^{x_1=\ell}\int_{x_2=0}^{x_2=\ell}\frac{(R-y_2\cos\theta)(x'+\beta R)Rf_x(x_1,y_1)f_x(x_2,y_2)\,dx_2dx_1dx'}{(R^2+y_2^2-2Ry_2\cos\theta)\sqrt{(x'+\beta R)^2-\beta^2(R^2+y_1^2-2Ry_1\cos\theta)}\ \sqrt{(x'+\beta R)^2-\beta^2(R^2+y_2^2-2Ry_2\cos\theta)}}\Bigg]d\theta dy_2dy_1 \tag{18}$$

This can be simplified because the second term within the brackets vanishes by virtue of the requirement

$$\int_{x_1=0}^{x_1=\ell}\int_{x_2=0}^{x_2=\ell}f_x(x_1,y_1)f_x(x_2,y_2)\,dx_2dx_1=0 \tag{19}$$

The expression for wave drag then reduces to

$$-4\pi^2\mathfrak{D}_I=\int_{y_1=-\infty}^{y_1=\infty}\int_{y_2=-\infty}^{y_2=\infty}\int_{\theta=0}^{\theta=2\pi}I\,d\theta dy_2dy_1 \tag{20}$$

where $a\gg\ell$ and

$$I=\int_{x'=0}^{x'=a}\int_{x_1=0}^{x_1=x'+\beta y_1\cos\theta;\ell}\int_{x_2=0}^{x_2=x'+\beta y_2\cos\theta;\ell}\frac{f_x(x_1,y_1)f_x(x_2,y_2)\,dx_2dx_1dx'}{2\sqrt{x'-x_1+\beta y_1\cos\theta}\ \sqrt{x'-x_2+\beta y_2\cos\theta}} \tag{21}$$

The domain of integration here is a region in $x'-x_1-x_2$ space whose cross section for $x'=\text{const}$ is a rectangle spanning $0\leq x_1\leq x'+\beta y_1\cos\theta$, $0\leq x_2\leq x'+\beta y_2\cos\theta$ for $x'\leq\ell$, and the square $0\leq(x_1,x_2)\leq\ell$ for $\ell<x'<a$. We can simplify Eq. (21) by interchanging the order of integration. To do this, we must have the proper limits, and these are obtained with the assistance of Fig. 2, depicting the volume in $x'-x_1-x_2$ space over which the summations are performed. Geometric considerations indicate that we can write equivalently $I=I_1+I_2+I_3+I_4$, where

$$I_1=\int_{x_2=0}^{x_2=\ell}\int_{x_1=0}^{x_1=\ell}\int_{x'=\ell}^{x'=a}\frac{f_x(x_1,y_1)f_x(x_2,y_2)\,dx'dx_1dx_2}{2\sqrt{x'-x_1+\beta y_1\cos\theta}\ \sqrt{x'-x_2+\beta y_2\cos\theta}} \tag{22}$$

refers to an integration over the lower base region in Fig. 2,

$$I_2=\int_{x_2=0}^{x_2=\beta y_2\cos\theta}\int_{x_1=0}^{x_1=\beta y_1\cos\theta}\int_{x'=0}^{x'=\ell}\frac{f_x(x_1,y_1)f_x(x_2,y_2)\,dx'dx_1dx_2}{2\sqrt{x'-x_1+\beta y_1\cos\theta}\sqrt{x'-x_2+\beta y_2\cos\theta}} \tag{23}$$

refers to the embedded parallelopiped in the upper pyramidal structure, and

$$I_3=\int_{x_2=\beta y_2\cos\theta}^{x_2=\ell+\beta y_2\cos\theta}\int_{x_1=0}^{x_1=x_2+\beta(y_1-y_2)\cos\theta}\int_{x'=x_2-\beta y_2\cos\theta}^{x'=\ell}\frac{f_x(x_1,y_1)f_x(x_2,y_2)\,dx'dx_1dx_2}{2\sqrt{x'-x_1+\beta y_1\cos\theta}\sqrt{x'-x_2+\beta y_2\cos\theta}} \tag{24}$$

and

$$I_4=\int_{x_1=\beta y_1\cos\theta}^{x_1=\ell+\beta y_1\cos\theta}\int_{x_2=0}^{x_2=x_1-\beta(y_1-y_2)\cos\theta}\int_{x'=x_1-\beta y_1\cos\theta}^{x'=\ell}\frac{f_x(x_1,y_1)f_x(x_2,y_2)\,dx'dx_2dx_1}{2\sqrt{x'-x_1+\beta y_1\cos\theta}\sqrt{x'-x_2+\beta y_2\cos\theta}} \tag{25}$$

refer to the remainder of the pyramidal solid. Each of the preceding terms can be simplified by applying the fact that

$$\int\frac{dx'}{\sqrt{(x'-\xi_1)(x'-\xi_2)}}=\ln[2x'-\xi_1-\xi_2+2\sqrt{(x'-\xi_1)(x'-\xi_2)}] \tag{26}$$

Equations (23-25) can be evaluated directly, leaving only double integrals, but Eq. (22) requires special treatment. Evaluation with Eq. (26) gives

$$I_1=\int_{x_2=0}^{x_2=\ell}\int_{x_1=0}^{x_1=\ell}\frac{f_x(x_1,y_1)f_x(x_2,y_2)}{2}\Phi dx_1dx_2 \tag{27}$$

where

$$\begin{aligned}\Phi=&\ln[2a-x_1-x_2+\beta y_1\cos\theta+\beta y_2\cos\theta+2\sqrt{(a-x_1+\beta y_1\cos\theta)(a-x_2+\beta y_2\cos\theta)}]\\&-\ln[2\ell-x_1-x_2+\beta y_1\cos\theta+\beta y_2\cos\theta+2\sqrt{(\ell-x_1+\beta y_1\cos\theta)(\ell-x_2+\beta y_2\cos\theta)}]\end{aligned} \tag{28}$$

Making use of the fact that $x_{1,2}-\beta y_{1,2}\cos\theta\sim O(\ell)$ and $a\gg\ell$, asymptotic expansion of the first term in Eq. (28) leads to

$$\Phi\simeq\ln 4a-\frac{x_1+x_2-\beta(y_1+y_2)\cos\theta}{2a}-\ln[2\ell-x_1-x_2+\beta y_1\cos\theta+\beta y_2\cos\theta+2\sqrt{(\ell-x_1+\beta y_1\cos\theta)(\ell-x_2+\beta y_2\cos\theta)}] \tag{29}$$

If this result is substituted into Eq. (27), the integral corresponding to $\ln 4a$ vanishes because of Eq. (19), and the integral corresponding to the second term in Eq. (29) vanishes as a tends to infinity, since it is $O(a^{-1})$. Thus we obtain, as the final result for the wave drag due to a planar distribution of sources,

$$D_{w_1}=\frac{\rho_\infty U_\infty^2}{4\pi^2}\int_{y_1=-\infty}^{y_1=\infty}\int_{y_2=-\infty}^{y_2=\infty}\int_{\theta=0}^{\theta=2\pi}I\,d\theta dy_2dy_1 \tag{30}$$

where I takes the form

$$\begin{aligned}I=&-\frac{1}{2}\int_{x_2=0}^{x_2=\ell}\int_{x_1=0}^{x_1=\ell}f_x(x_1,y_1)f_x(x_2,y_2)\ln[2\ell-x_1-x_2+\beta y_1\cos\theta+\beta y_2\cos\theta+2\sqrt{(\ell-x_1+\beta y_1\cos\theta)(\ell-x_2+\beta y_2\cos\theta)}]dx_1dx_2\\
&+\int_{x_2=0}^{x_2=\beta y_2\cos\theta}\int_{x_1=0}^{x_1=\beta y_1\cos\theta}\frac{f_x(x_1,y_1)f_x(x_2,y_2)}{2}\left\{\begin{array}{l}\ln[2\ell-x_1-x_2+\beta y_1\cos\theta+\beta y_2\cos\theta\\+2\sqrt{(\ell-x_1+\beta y_1\cos\theta)(\ell-x_2+\beta y_2\cos\theta)}]\\-\ln[-x_1-x_2+\beta y_1\cos\theta+\beta y_2\cos\theta\\+2\sqrt{(x_1-\beta y_1\cos\theta)(x_2-\beta y_2\cos\theta)}]\end{array}\right\}dx_1dx_2\\
&+\int_{x_2=\beta y_2\cos\theta}^{x_2=\ell+\beta y_2\cos\theta}\int_{x_1=0}^{x_1=x_2+\beta(y_1-y_2)\cos\theta}\frac{f_x(x_1,y_1)f_x(x_2,y_2)}{2}\left\{\begin{array}{l}\ln[2\ell-x_1-x_2+\beta y_1\cos\theta+\beta y_2\cos\theta\\+2\sqrt{(\ell-x_1+\beta y_1\cos\theta)(\ell-x_2+\beta y_2\cos\theta)}]\\-\ln[x_2-x_1-\beta y_2\cos\theta+\beta y_1\cos\theta]\end{array}\right\}dx_1dx_2\\
&+\int_{x_1=\beta y_1\cos\theta}^{x_1=\ell+\beta y_1\cos\theta}\int_{x_2=0}^{x_2=x_1-\beta(y_1-y_2)\cos\theta}\frac{f_x(x_1,y_1)f_x(x_2,y_2)}{2}\left\{\begin{array}{l}\ln[2\ell-x_1-x_2+\beta y_1\cos\theta+\beta y_2\cos\theta\\+2\sqrt{(\ell-x_1+\beta y_1\cos\theta)(\ell-x_2+\beta y_2\cos\theta)}]\\-\ln[x_1-x_2-\beta y_1\cos\theta+\beta y_2\cos\theta]\end{array}\right\}dx_2dx_1\end{aligned} \tag{31}$$

Next we consider the drag contribution due to doublets alone, represented by the potential

$$\begin{aligned}\phi_{D(z)}&=-\frac{1}{2\pi}\frac{\partial}{\partial z}\int_{y_1=-\infty}^{y_1=\infty}\int_{x_1=-\infty}^{x_1=x-\beta\sqrt{r^2+y_1^2-2ry_1\cos\theta}}\frac{P(x_1,y_1)\,dx_1dy_1}{\sqrt{(x-x_1)^2-\beta^2(r^2+y_1^2-2ry_1\cos\theta)}}\\
&=\frac{1}{2\pi}\int_{y_1=-\infty}^{y_1=\infty}\int_{x_1=-\infty}^{x_1=x-\beta\sqrt{r^2+y_1^2-2ry_1\cos\theta}}\frac{r\sin\theta(x-x_1)P_{x_1}(x_1,y_1)\,dx_1dy_1}{(r^2+y_1^2-2ry_1\cos\theta)\sqrt{(x-x_1)^2-\beta^2(r^2+y_1^2-2ry_1\cos\theta)}}\end{aligned} \tag{32}$$

To calculate the required derivatives, $P(-\infty, y_1) = P_{x_1}(-\infty, y_1) = 0$ is assumed in the preliminary integration by parts, eliminating the appearance of singular integrals. It will be convenient to denote by λ the linear integral operator in Eq. (31), that is,

$$I(\theta, y_2, y_1; \ell, \beta) = \lambda\{f_x(x_1, y_1) f_x(x_2, y_2)\} \tag{33}$$

Carrying through an analysis similar to the foregoing leads to the result that

$$D_{w_2} = \frac{\rho_\infty U_\infty^2}{4\pi^2} \int_{y_1=-\infty}^{y_1=\infty} \int_{y_2=-\infty}^{y_2=\infty} \int_{\theta=0}^{\theta=2\pi} \lambda\{\beta^2 \sin^2\theta P_{xx}(x_1, y_1) P_{xx}(x_2, y_2)\} \mathrm{d}\theta \mathrm{d}y_2 \mathrm{d}y_1 \tag{34}$$

The "interaction drags" $\mathfrak{D}_3$ and $\mathfrak{D}_4$ can be calculated similarly, leading to the result that the total wave drag D_w is expressible in the form

$$D_w = \frac{\rho_\infty U_\infty^2}{4\pi^2} \int_{y_1=-\infty}^{y_1=\infty} \int_{y_2=-\infty}^{y_2=\infty} \int_{\theta=0}^{\theta=2\pi} \lambda\{f_x(x_1, y_1) f_x(x_2, y_2) + \beta^2 \sin^2\theta P_{xx}(x_1, y_1) P_{xx}(x_2, y_2)$$
$$- 2\beta \sin\theta P_{xx}(x_2, y_2) f_x(x_1, y_1)\} \mathrm{d}\theta \mathrm{d}y_2 \mathrm{d}y_1 \tag{35}$$

One can show further that[6] the relation

$$\lambda\{\ \} = \lambda\{[f_x(x_1, y_1) - \beta \sin\theta P_{xx}(x_1, y_1)][f_x(x_2, y_2) - \beta \sin\theta P_{xx}(x_2, y_2)]\}$$

holds, and that it follows from a type of nonlinear reciprocity relation. The appearance of this product was suggested by Eq. (4). However, it is not the lift density ℓ that belongs in the exact expression but the doublet derivative P_x.

It is interesting to see how our results reduce to classical ones in one particular example. We calculate the drag due to a lineal distribution of sources by dropping the iterated integrals over y_1 and y_2 and setting $y_1 = y_2 = 0$ throughout. In the fourth integral of Eq. (31), we exchange x_1 for x_2 and x_2 for x_1. Addition then produces the θ-independent expression

$$I = -\int_{x_2=0}^{x_2=\ell} \int_{x_1=0}^{x_1=\ell} \frac{f_x(x_1) f_x(x_2)}{2} \ln[2\ell - x_1 - x_2 + 2\sqrt{(\ell - x_1)(\ell - x_2)}] \mathrm{d}x_1 \mathrm{d}x_2$$
$$+ \int_{x_2=0}^{x_2=\ell} \int_{x_1=0}^{x_1=x_2} f_x(x_1) f_x(x_2) \{\ln[2\ell - x_1 - x_2 + 2\sqrt{(\ell - x_1)(\ell - x_2)}] - \ln[x_2 - x_1]\} \mathrm{d}x_1 \mathrm{d}x_2 \tag{36}$$

The first multiple integral is canceled by the first term of the second, since the integrand is symmetric about the line $x_1 = x_2$, leaving

$$I = -\int_{x_2=0}^{x_2=\ell} \int_{x_1=0}^{x_1=x_2} f_x(x_1) f_x(x_2) \ln|x_2 - x_1| \mathrm{d}x_1 \mathrm{d}x_2 = -\frac{1}{2} \int_{x_2=0}^{x_2=\ell} \int_{x_1=0}^{x_1=\ell} f_x(x_1) f_x(x_2) \ln|x_2 - x_1| \mathrm{d}x_1 \mathrm{d}x_2 \tag{37}$$

for which

$$D_w = \frac{\rho_\infty U_\infty^2}{4\pi^2} \int_{\theta=0}^{\theta=2\pi} I \mathrm{d}\theta = -\frac{\rho_\infty U_\infty^2}{2\pi} \int_{x_2=0}^{x_2=\ell} \int_{x_1=0}^{x_1=x_2} f_x(x_1) f_x(x_2) \ln|x_2 - x_1| \mathrm{d}x_1 \mathrm{d}x_2$$
$$= -\frac{\rho_\infty U_\infty^2}{4\pi} \int_{x_2=0}^{x_2=\ell} \int_{x_1=0}^{x_1=\ell} f_x(x_1) f_x(x_2) \ln|x_2 - x_1| \mathrm{d}x_1 \mathrm{d}x_2 \tag{38}$$

This reproduces von Kármán's[1] result for lineal source distribution.

The drag formula given in Eq. (35) is an exact result of linearized supersonic flow and does not involve any approximations except that in Eq. (19). For the case where the maximum cross-dimension is small compared to ℓ, say, it is

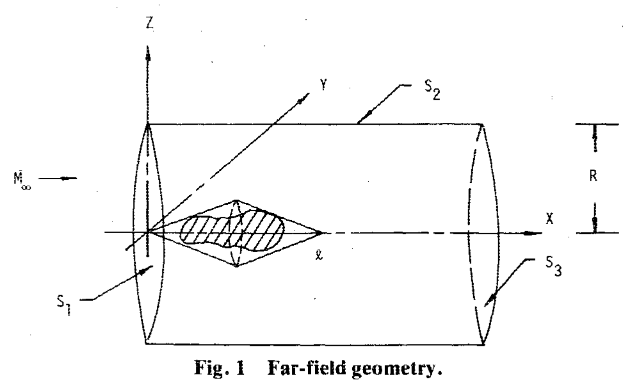

Fig. 1 Far-field geometry.

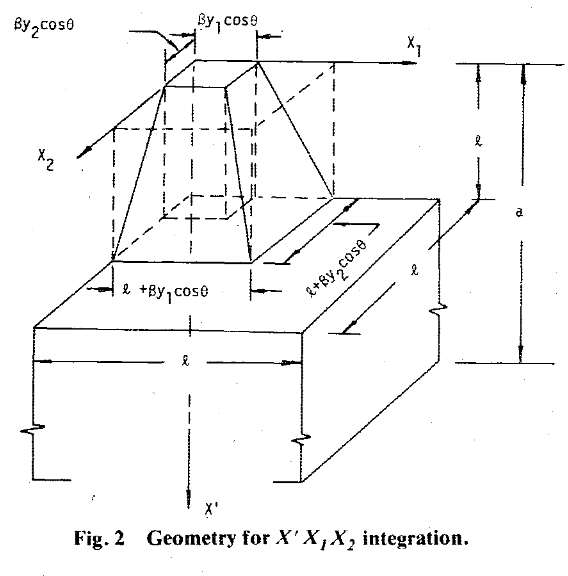

Fig. 2 Geometry for $X' X_1 X_2$ integration.

possible to show[6] that the interaction drags are small in comparison with those due to sources and doublets, and this is expected from slender-body theory. It is apparent from the foregoing discussion that drag theorems for the reversed flowfield can be derived without difficulty and without conceptual change. However, the results for the combined flowfield (that is, the sum of forward and reverse drags) are no longer as elegant as those for near-planar bodies.

Concluding Remarks

We can make several summarizing statements bringing our results in perspective with Hayes.[1] The point of our analysis is this: the most important restriction in Hayes' theory is the implied near-planarity. This arises in considering the far-field momentum flux due to distributions of horseshoe vortices, which are restricted to lie along surfaces whose generators are aligned with the streamwise axis of the governing differential equation; the classical results therefore bear the limiting relationships connecting, for example, local vorticity strength to local lift, and fail to account properly for lift and thickness interaction. Hayes implicitly assumes a planar vortex sheet behind each lifting element, whereas a doublet formulation allows the singularity surface to be nonplanar.

A second point concerns the source drag integrand I formally given in Eq. (21) and expanded out in Eq. (31); the operator associated with I enters in the more complete drag formula that includes the combined effects of lift and lift/thickness interaction [see Eqs. (33) and 35)]. In the form given by Eq. (21) or Eq. (31), the evaluation of I is straightforward, although tedious; the purpose of this evaluation is to show how the complete drag formula can be written down explicitly, and the availability of this result is desirable and useful in many direct applications. Of course, Eq. (21) could be simplified considerably (symbolically, though) following Hayes' example of lumping the singularities on the x axis by integrating them along the Mach planes, and this can be done easily. The results of our very tedious asymptotic evaluation indicate that it is not the lift density ℓ that belongs in Hayes' drag integral but the doublet derivative. (These two are equal only for planar bodies.) This suggests that Hayes' method [summarized by Eqs. (5, 2, and 3), as previously described] should be modified by replacing the classical use of Eq. (4) by $\tilde{h}=\tilde{f}-\beta P_{xl}\sin\theta$; the drag integral with the operator I as defined in Eq. (21) or (31) is therefore completely equivalent to using Hayes' integral formula (for singularities lumped along the x axis) but using the foregoing $\tilde{h}$, and this should afford some computational advantage. This result is part of a more general one derived in Ref. 2 using a different but complementary approach; Eq. (8), which is valid for arbitrary surfaces, reduces in the planar limit (that is, using $Z_l=0$ and $G_l=1$) to $\tilde{h}=\tilde{f}-R_{xl}-\beta P_{xl}\sin\theta-\beta Q_{xl}\cos\theta$, which implicitly contains the present results. [Equation (8) furthermore shows the nontrivial effect of streamwise surface slope and curvature.] The present paper considers in detail the structure of the drag integral, provides an explicit evaluation of the "equivalent source," and confirms the more general results of Ref. 2.

The use of exact, explicit formulas such as those given in Eqs. (35) and (8) appears in both supersonic design and analysis problems. Aside from their direct value to wave drag calculations, they reduce the labor necessary to compute incremental changes in D_w due to incremental changes in singularity strength or geometry, or vice-versa. For these applications, the basic linearization would be carried out about a given nonplanar geometry, say. Additional uses also can be found. Advanced design concepts presently pursued by the author involve the application of these exact formulas in wave drag minimization; variational calculus is used to determine optimal geometries under various physical constraints. Other uses no doubt, will appear, and various possibilities currently are being explored.

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